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TARCMO: Theory and Algorithms for Robust, Combinatorial, Multicriteria Optimization

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14. ABSTRACT
This project has completed. The PI considered optimization problems with uncertainty in the data. Specifically, algorithms and analysis methods were developed for several optimization scenarios where the random parameters were coming from some unknown probability distribution. One notable result efficiently calculates the average optimal solution for a combinatorial regret problem and provides new bounds for how for this may be from the worst case. For details, see the final report. Further, the project produced 8 published papers and 4 more under review or preparation. The specifics of these references are included in uploaded final report. This work has help to advance the state-of-the-art in robust optimization.

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Contents

1 Summary

The general objectives of TARCMO are

a) to collect and design scalable algorithms for robust optimization

b) to find evaluation schemes for robust solutions

c) to provide a software tool that unifies these algorithms and evaluation schemes

d) to apply these results to network flow and location theory problems

e) to generalize these results to multi-criteria optimization.

In the following, we summarize the progress that has been made on these objectives over the whole funding period.

2 Introduction

Robust optimization is an active field of research that considers optimization problems under uncertainty. Contrary to the similar setting of *stochastic optimization*, a probability distribution is usually not required. Recent surveys on the topic include [GS16, BTGN09, BBC11].

The research conducted in this project aims at improving the applicability of robust optimization. This includes several aspects: In most cases, a robust optimization problem is significantly harder to solve than its non-robust counterpart. We therefore develop algorithms that scale well with the problem size. Furthermore, there are currently no means to assess the quality of a robust solution, and hence, to guide a practitioner which kind of robustness approach is best-suited for his needs. To this end, we develop evaluation schemes that help comparing robust and non-robust solutions.

3 Methods, Assumptions, and Procedures

We consider general optimization problems of the form

$$\begin{aligned} \min\ & f(x,\xi) \\ \text{s.t. } & F(x,\xi) \le 0 \\ & x \in \mathcal{X}, \end{aligned}$$

where the uncertain data is represented by the scenario parameter ξ coming from an uncertainty set $\mathcal{U}$. No probability distribution over $\mathcal{U}$ is known. We denote the set of feasible solutions in scenario $\xi \in \mathcal{U}$ as $\mathcal{F}(\xi)$.

In robust optimization there exist several concurrent approaches how to reformulate such a (possibly infinite) family of problems to one single robust counterpart. We note three such approaches here (for an overview on other counterparts, see [GS16]).

- Strict robustness [BTN98] considers counterparts of the form

$$\begin{aligned} \min \ & \max_{\xi \in \mathcal{U}} f(x, \xi) \\ \text{s.t. } & F(x, \xi) \leq 0 \ \forall \xi \in \mathcal{U} \\ & x \in \mathcal{X}, \end{aligned}$$

- The approach of *regret robustness* [ABV09] aims at finding robust solutions which perform well in every scenario, compared to the best possible performance that could be achieved in each scenario, i.e., regret robustness is represented with the following program:

$$\begin{aligned} \min \ & \max_{\xi \in \mathcal{U}} f(x, \xi) - f^*(\xi) \\ \text{s.t. } & F(x, \xi) \leq 0 \ \forall \xi \in \mathcal{U} \\ & x \in \mathcal{X}, \end{aligned}$$

  where $f^*(\xi) := \min\{f(y, \xi) : F(y, \xi) \leq 0 \ \forall \xi \in \mathcal{U}, \ y \in \mathcal{X}\}$. Often, uncertainty is only considered in the objective function, and not in the constraints.

- For recovery robustness [LLMS09], one assumes the existence of a *recovery algorithm* which allows the modification of a solution once the scenario becomes known. It can therefore be considered as a two-stage approach to robust optimization.

Typical uncertainty sets include

- finite uncertainty of the form

$$\mathcal{U} = \left\{\xi^1, \ldots, \xi^N\right\}$$

- interval-based uncertainty

$$\mathcal{U} = [\underline{\xi}_1, \overline{\xi}_1] \times \ldots \times [\underline{\xi}_M, \overline{\xi}_M]$$

- general ellipsoidal uncertainty

$$\mathcal{U} = \{A\xi + a_0 : ||\xi|| \leq 1\}$$

- axis-parallel ellipsoidal uncertainty

$$\mathcal{U} = \left\{ \xi \in \mathbb{R}^M : \sum_{i=1}^{M} d_i^2 (\xi_i - \xi_i^0)^2 \leq 1 \right\}$$

- Bertsimas-Sim-type uncertainty

$$\mathcal{U} = \left\{ \xi \in \mathbb{R}^M : \xi_i \in [\hat{\xi}_i - d_i, \hat{\xi}_i + d_i], \sum_{i=1}^{M} \frac{|\xi_i - \hat{\xi}_i|}{d_i} \leq \Gamma \right\}$$

4 Results and Discussion

In the following, we summarize main research topics and results of the project.

4.1 Bounds for General Combinatorial Regret Problems.

We first describe our results on the analysis of the midpoint solution x_{mid} for regret problems with interval uncertainty, where the underlying nominal problem is combinatorial (i.e., all variables are binary). The midpoint solution can be found by solving a nominal problem with respect to the midpoint (i.e., average) of the uncertainty. We were able to show that this is a λ-approximation, where λ can be computed as

$$\lambda = \frac{val(x_{mid}, c^{\mathbf{x}_{mid}}) - val^*_{c^{\mathbf{x}_{mid}}}}{val(x_{mid}, \hat{c}) - \min_{S \subseteq \mathcal{N}} \frac{1}{2}(val^*_{c^S} + val^*_{c^{\overline{S}}})}$$

and is always less our equal to 2, meaning that this bound is always at least as good as the current best known bound.

To compute this bound, an auxiliary optimization problem needs to be solved. We were able to show that this problem can be solved in strongly polynomial time for the shortest path problem, the minimum spanning tree problem, the assignment problem, and the minimum $s - t$ cut problem.

This lower bound is a valuable, as it allows to estimate how good the midpoint solution performs. Additionally, this lower bound can also be used as part of a branch-and-bound approach to find an even optimal solution to the regret problem.

To analyze the computational improvement when using this bound, we considered the regret shortest path problem. Here, the current best performance of a branch-and-bound algorithm was given in [MGD04]. Following the same algorithm on equally generated instances using our new bound, we considerably improved computation times by approximately an order of magnitude. Figure 1 shows a representative run of both algorithms.

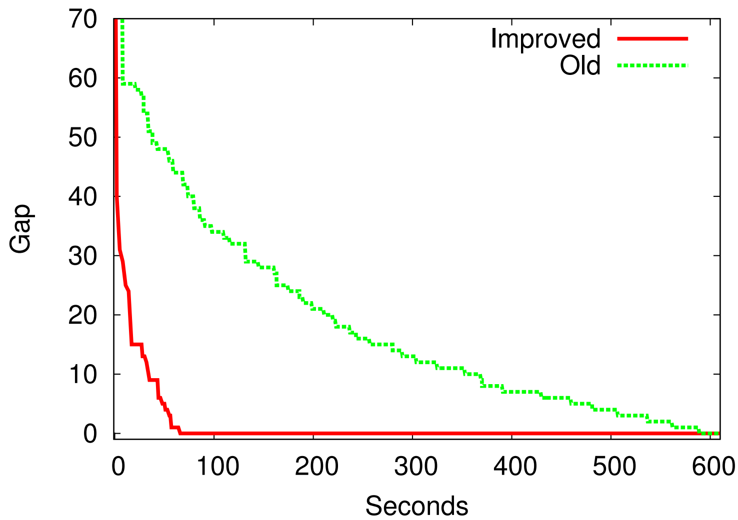


Figure 1: Example run comparing our new bound (red) to the former best approach (green).

The "gap" is the difference between the objective value of the current best solution, and the best lower bound in each iteration. When the gap is zero, the instance is solved to provable optimality. While the former approach takes about 10 minutes of computation time, the new approach needs only one minute.

The research on this topic has been published as [CG15b], where details can be found.

4.2 Bounds for the Regret Knapsack Problem.

We further analyzed the application to the knapsack problem as a special case of regret robustness with a finite uncertainty set. The current best solution method is a branch-and-bound approach using a surrogate relaxation bound [Iid99, TYK08].

We developed a new upper bound for the knapsack problem, which is specifically designed to counter the drawbacks of the surrogate bound. While the latter performs well at early nodes in the branch-and-bound tree, the new bound shows

best performance for deep nodes in the tree. Combining both bounds results in computation times an order of magnitude better than before.

As an example, we found that for instances with 200 uncorrelated scenarios and 35 items, the surrogate bound approach takes on average 61 seconds to find an optimal solution. With our improved bound, this computation time could be reduced to only 4 seconds on average.

As the surrogate bound is a technique that can be applied to a large class of problems, our results can potentially be extended to these as well.

The research on this topic has been published as [Goe14], where details can be found.

4.3 On the Evaluation of Solutions.

As has been explained in the project application, current methods to assess the quality of a robust solution like the price of robustness [BS04, MP13] have their drawbacks. We developed alternative ways to evaluate solutions distinguishing between two cases: *(a)* The uncertainty is only in the objective function, and not in the constraints. *(b)* The uncertainty is both in the objective function and in the constraints.

Concerning case *(a)*, we propose the usage of the *scenario curve*, which is generated in the following way. Given a (robust) solution, we calculate its objective value in each scenario, and sort these values from their best to their worst. Doing so for all solutions under consideration makes sure that the performance of a solution is always compared to a performance of similar quality of another solution. Figure 2 shows an example scenario curve for a knapsack problem.

This approach gives rise to a new robustness concept, that aims at optimization the performance for a given number of scenarios, which we call n-case robustness. It allows the practitioner to directly control the degree of worst-case quality he likes to achieve.

For the case *(b)* where also constraints are affected by uncertainty, we cannot simply calculate the objective value of a solution in each scenario, as it may become infeasible. Thus, we assume the existence of a *recovery algorithm* that allows the modification of a given solution to become feasible for a given scenario. Such a recovery procedure has a cost, and the available recovery budget determines the objective value that is achievable for a scenario.

This allows to calculate a scenario curve for every possible fixed recovery budget. Figure 3 gives an example plot for the midpoint solution (left) and the strictly robust solution (right) for an uncertain knapsack problem.

The horizontal axis represents the available recovery budget, and the vertical axis the respective scenario curve. All values are normalized with respect to

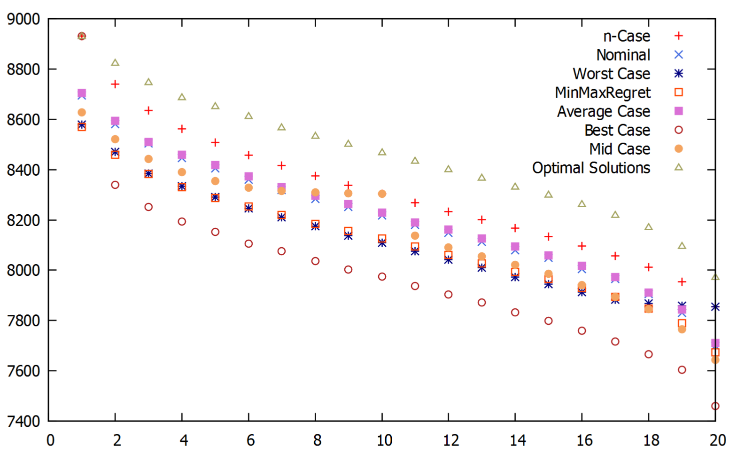
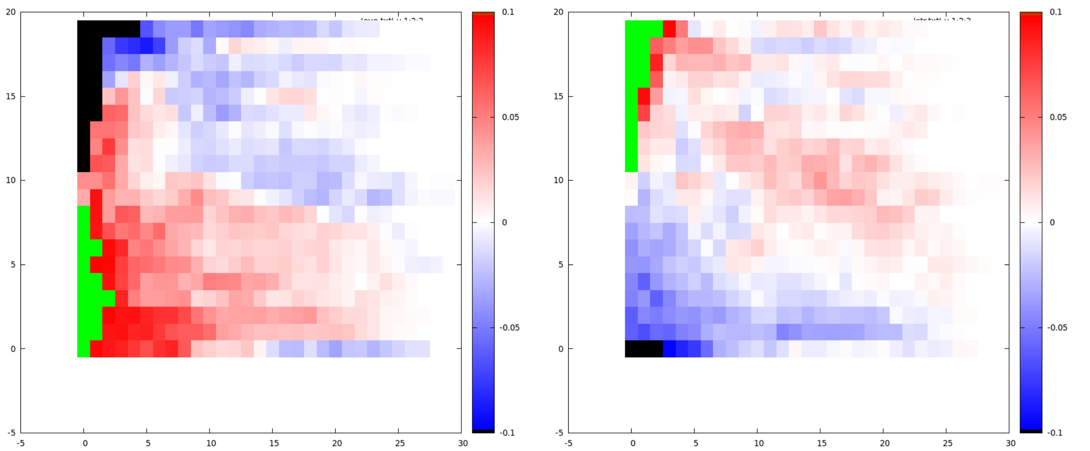


Figure 2: Example performance analysis for optimization problems with uncertainty the objective.



(a) Performance of the midpoint solution. (b) Performance of the strictly robust solution.

Figure 3: Example performance analysis for optimization problems with uncertainty in both objective and constraints.

the average of several considered solutions, where red are higher than average objective values, and blue are lower than average objective values. Green and black are especially good (or bad) values, respectively. As can be seen, the strictly robust solution has a very good performance for the worst-performing scenarios, even with low recovery budget (upper left corner), while for the midpoint solution, this is mostly the other way around.
A summary of these methods is presented in the book chapter [CG16d].

4.4 Robust Timetable Information Problems.

Timetable information is the process of determining a suitable travel route for a passenger in public transport, especially railway systems. Due to delays in the original timetable, in practice it often happens that the travel route cannot be used as originally planned. For a passenger being already en route, it would hence be useful to know about alternatives that ensure that his/her destination can be reached.

We proposed a recoverable robust approach to timetable information; i.e., we aim at finding travel routes that can easily be updated when delays occur during the journey. We developed polynomial-time algorithms for this problem and evaluated the performance of the routes obtained this way on schedule data of the German train network of 2013 and simulated delay scenarios.

We found that this new approach to the timetable information is able to find paths that perform considerably better than other approaches in terms of travel times. In Figure 4 we give a histogram showing the improvement of the recoverable robust approach over the standard timetable information method.

The research on this topic has been published as [GHMH$^+$13], where details can be found.

4.5 Alternative formulations for the ordered weighted averaging objective

The ordered weighted averaging (OWA) objective has been developed in a multi-criteria context, but can be regarded as a direct generalization of the classic min-max worst-case approach in robust optimization. Starting from a linear program with multiple objective function (e.g., a different objective function per scenario) of the form

$$\max \left\{ Cx \mid Ax = b,\ x \in \mathbb{R}^n_+ \right\}$$

with $C \in \mathbb{R}^{k\times n}$, we use an ordering map $\Theta : \mathbb{R}^k \to \mathbb{R}^k$ that sorts a vector increasingly, i.e., $\theta_i(y) \leq \theta_{i+1}(y)$ for $i = 1,\dots,k-1$. In the OWA setting, one

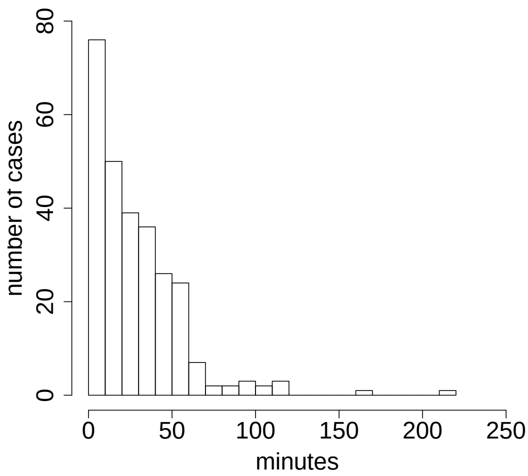


Figure 4: Histogram for the number of cases where we can save x minutes by choosing a recoverable robust path instead of the standard path.

now considers the aggregated objective function

$$\max \left\{ \sum_{i=1}^{k} w_i \theta_i(Cx) \mid Ax = b,\ x \in \mathbb{R}^n_+ \right\}$$

for some weights w_i. In other words, for $w_1 = w_2 = \ldots = w_{n-1} = 0$ and $w_n = 1$, this becomes the robust worst-case objective, while for $w_i = 1/k$ for all $i = 1, \ldots, k$, the objective is the arithmetic mean over all scenarios.

Objective functions of this type have seen recently rising interest in the research community, due to their very general nature. However, they may be difficult to solve. A reformulation approach of the OWA objective under mild additional assumptions has been published in [OŚ03], where the following program is proposed:

$$\begin{aligned}
\max\ & \sum_{j\in[k]} j w'_j r_j - \sum_{i\in[k]} \sum_{j\in[k]} w'_j d_{ij} \\
\text{s.t. } & Cx = y \\
& d_{ij} \geq r_j - y_i && \forall i, j \in [k] \\
& Ax = b \\
& x \in \mathbb{R}^n_+, d \in \mathbb{R}^{k\times k}_+, y \in \mathbb{R}^k, r \in \mathbb{R}^k
\end{aligned}$$

By rewriting the OWA objective using the set of all possible permutations of a vector, and using duality techniques, we were able to find a new and improved

formulation of the following form:

$$
\begin{aligned}
\max\ & \sum_{i\in[k]}(\alpha_i+\beta_i) \\
\text{s.t. } & Cx = y \\
& \alpha_i + \beta_j \le w_i y_j && \forall i,j\in[k] \\
& Ax = b \\
& x\in\mathbb{R}^n_+, \alpha,\beta,y\in\mathbb{R}^k
\end{aligned}
$$

This formulation is not only more elegant and uses less variables, but also shows considerably improved solution times by an order of magnitude in computational experiments on portfolio optimization problems.

The research on this topic has been published as [CG15a], where details can be found.

4.6 On The Recoverable Robust Traveling Salesman Problem

The traveling salesman problem (TSP) is a well-known combinatorial optimization problem, which asks for a circular tour visiting a set of destinations (nodes) with minimal tour length (see, e.g., [Coo12]). Applications can be found, e.g., in circuit board drilling, computer wiring, vehicle routing, and many more real-world problems.

We considered an uncertain variant of this problem, where distances between nodes are not known exactly, but stem from either a finite uncertainty set, or an uncertainty of the Bertsimas and Sim type. To find robust traveling salesman tours, we followed a recoverable approach; meaning that once the scenario becomes known, we are allowed to update the current solution. These updates are bounded by the number of direct connections that we change. The resulting robust problem is a min-max-min problem, and therefore not solvable directly.

We developed an iterative solution procedure, where three subproblems are connected: (P1), given a finite set of scenarios, find a robust tour and an updated tour for every scenario, such that the worst-case travel time is minimized. (P2), given a robust tour and a set of alternative tours, find a worst-case scenario maximizing the travel time. (P3), give a scenario and a robust tour, find the best possible updated tour. The interplay of these problems is visualized in Figure 5. In the left hand of the figure, we see iterations between P1, which produces a new candidate solution, and P2+P3, which evaluate this solution. Lower and upper bounds produced this way converge to an optimal solution after 8 iterations. In the right hand of the figure, we see the evaluation process of a solution, which is in itself again an iterative process using P2 and P3.

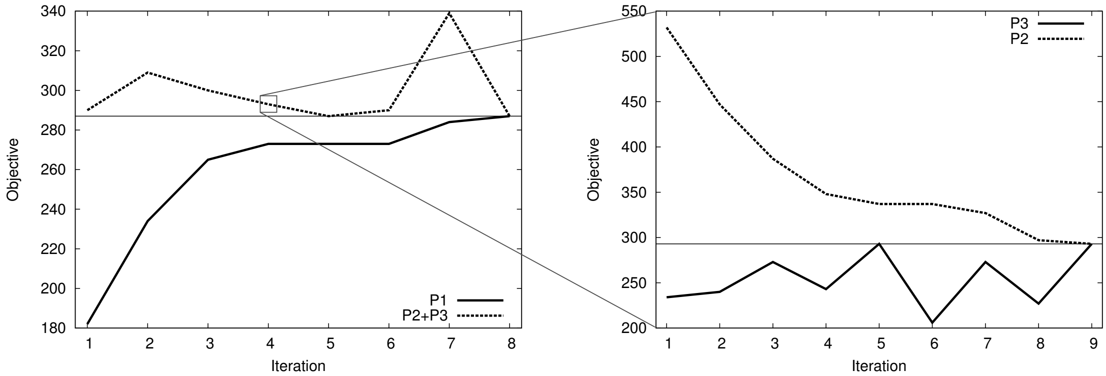


Figure 5: The iterative solution procedure for the robust traveling salesman problem.

While this iterative algorithms results in an optimal solution to the robust TSP, computation times become too high for larger instances. For this purpose, we developed a local search meta-heuristic that uses a simplified evaluation scheme for robust solutions. To this end, we integrate problems P2 and P3 using duality on the linear programming relaxation of P3. This location search heuristic shows excellent performance also on larger uncertain instances.

The research on this topic has been published as [CG16c], where details can be found.

4.7 A Bicriteria Approach to Robust Optimization

We developed a new evaluation scheme for linear programs with an uncertain objective function. The uncertainty is represented in our model either by a finite set of scenarios or by box uncertainty. For the new scheme we consider the nominal optimization problem that has only one criterion as a bicriteria optimization problem. The first criterion is the performance in the average case scenario (AC) and the second criterion is the worst case guarantee (WC) for all scenarios.

We solve the bicriteria problem by computing the corresponding Pareto front. We denote this Pareto front as the AC–WC curve (see Figure 6 for some examples of AC–WC curves). Further, we developed a decomposition method that uses column generation to compute the AC–WC curve. This method proves to be effective in computational experiments, especially if we can use specialized algorithms to solve the nominal problem. Network flow problems are a typical example for which such specialized algorithms exist, that outperform linear programming solvers. For a comparison of computation times between the straightforward approach to calculate the AC–WC curveand our decomposition method, see Figure 7.

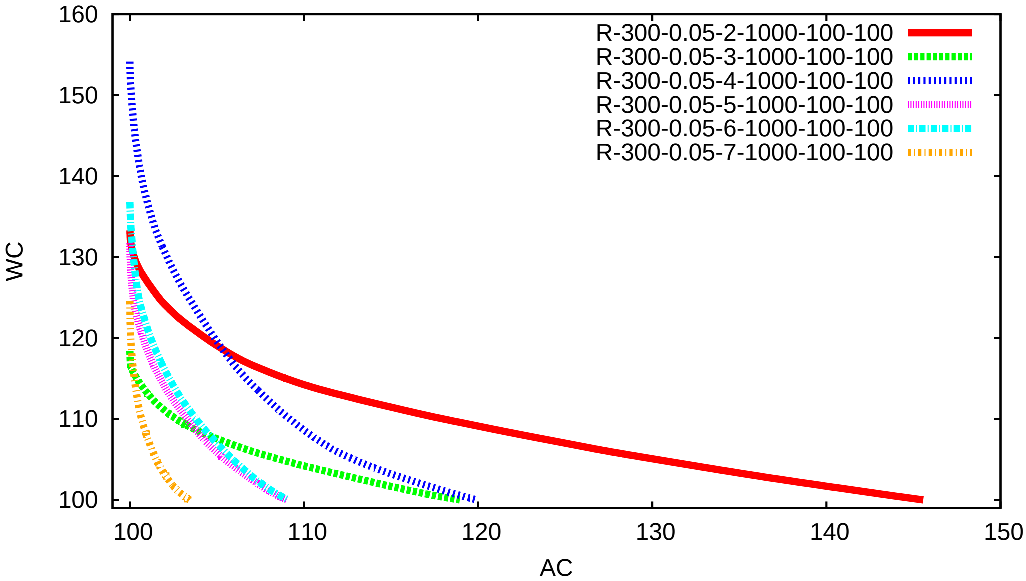


Figure 6: The AC–WC curves of different network flow problems with 300 nodes and 2-7 different cost scenarios. The average and worst case performances are normalized to 100.

As the Pareto front of multicriteria optimization problems might contain exponentially many efficient points and might, therefore, be too large to be computed exactly, we investigated an approximation algorithm that synergizes well with the proposed column generation method. For the tested instances the approximation algorithm is able to produce good approximations of the AC–WC curve in only 5-10–times the time that is needed to compute only the worst case solution.

In the literature (see, e.g., [BS04]) it is a common approach to ignore the uncertainty in the objective function for linear programs, as the linear objective function can be represented by a constraint, and to use concepts that are applicable if the constraints of the linear program are uncertain. We show that this approach has some drawbacks by comparing solutions that are generated with well known robustness concepts with the AC–WC curve. We denote the concept that was introduced by Ben-Tal and Nemirovski ([BTN99]) with Ω-robustness and the concept of Bertsimas and Sim ([BS04]) by Γ-robustness. Both concepts contain a parameter $(\Gamma \backslash \Omega)$ that can be used to control the conservatism of the computed solution. For extreme values these concepts produce the worst case

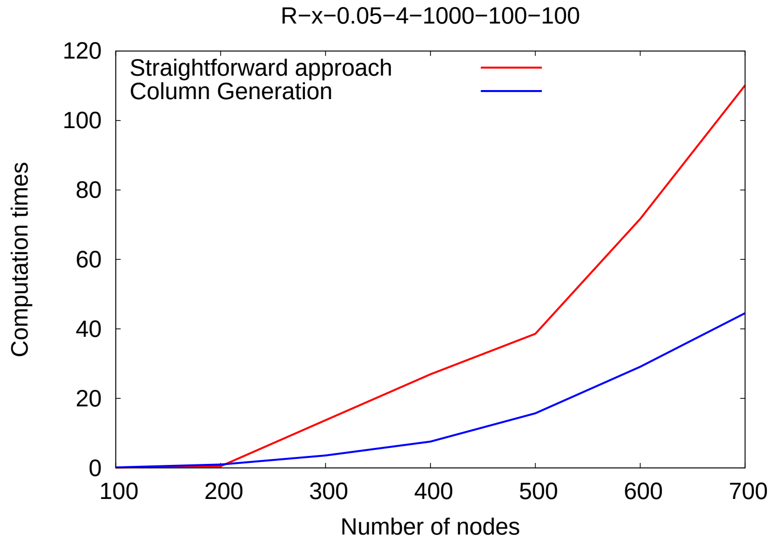


Figure 7: Time needed to compute the AC–WC curve. The straightforward approach does not use the decomposition technique and relies purely on the solution of linear programs.

and average case solution but in between they differ significant from the AC–WC curve (see Figure 8).

The research on this topic has been published as [CG16a], where details can be found.

4.8 Robust Geometric Programming is co-NP hard

In a more theoretical study we investigated the complexity status of robust geometric programming. Geometric Programming is a useful tool with a wide range of applications in engineering. As in real-world engineering problems input data is likely to be affected by uncertainty, Hsiung, Kim, and Boyd ([HKB08]) introduced robust geometric programming to include the uncertainty in the optimization process. They also developed a tractable approximation method to tackle this problem. Furthermore, they pose the question whether there exists a tractable reformulation of their robust geometric programming model instead of only an approximation method. We give a negative answer to this theoretical question by showing that robust geometric programming is co-NP hard in its natural posynomial form. A preprint of the material is available as [CG14].

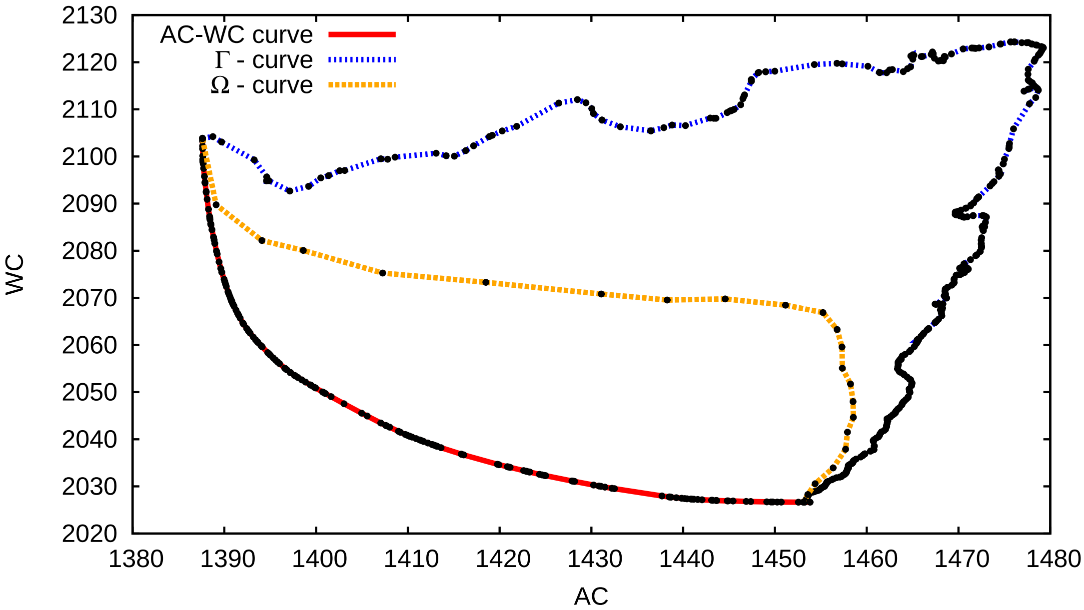


Figure 8: The $\Gamma\backslash\Omega$-curves are obtained by solving the corresponding $\Gamma\backslash\Omega$-robust counterparts for different values of $(\Gamma\backslash\Omega)$. The underlying problem is a network flow problem with 400 nodes. The costs for every arc are assumed to belong to certain intervals.

4.9 The Quadratic Shortest Path Problem: Complexity, Approximability, and Solution Methods

We considered the problem of finding a shortest path in a directed graph with a quadratic objective function (the QSPP). This problem is closely related to the robust shortest path problem with ellipsoidal uncertainty. The robust shortest path problem can be solved by solving a sequence of QSPPs. We showed that the QSPP cannot be approximated unless $\mathsf{P} = \mathsf{NP}$. For the case of a convex objective function, an n-approximation algorithm was found, where n is the number of nodes in the graph, and APX-hardness is shown. Furthermore, we proved that even if only adjacent arcs play a part in the quadratic objective function, the problem still cannot be approximated unless $\mathsf{P} = \mathsf{NP}$.

Beside the theoretical analysis we also tried to find exact solution methods for the QSPP. First we defined a mixed integer programming formulation, and then devise an efficient exact Branch-and-Bound algorithm for the general QSPP, where lower bounds are computed by considering a reformulation scheme that is solvable through a number of minimum cost flow problems. In our computational experiments we solved to optimality different classes of instances with up to 1000

nodes. The result of one experiment is reported in Table 1. The underlying network of this experiment are grid networks with n rows and m columns. We compared three different algorithmic approaches. Cplex(QP) is the naive approach to use the standard quadratic integer programming formulation of QSPP and solve it with Cplex, Cplex(MILP) uses an improved mixed integer linear programming formulation and B-and-B represents the Branch and Bound algorithm we developed.

Instance			Cplex (QP)			Cplex (MILP)			B-and-B		
n	m	opt.	lb_{root}	nodes	time(s)	lb_{root}	nodes	time(s)	lb_{root}	nodes	time(s)
258	512	622	-330.0	9747	1951.3	530.6	161	4.3	593.6	89	3.9
258	512	632	-333.1	10599	2357.2	530.8	235	5.6	588.9	123	4.2
258	512	650	-334.7	14249	2866.3	530.6	309	6.4	564.6	99	3.8
258	512	641	-333.9	13720	1525.7	514.5	295	5.7	586.0	91	4.3
258	512	593	-329.6	8533	1749.5	521.9	74	3.7	562.8	49	3.6
531	1058	1283	-759.4	4684	TL	997.9	5579	518.6	1125.6	414	22.1
531	1058	1281	-757.0	4783	TL	1001.3	4899	492.9	1146.4	438	22.2
531	1058	1302	-812.7	4688	TL	1007.4	4944	490.1	1130.3	768	25.5
531	1058	1283	-757.8	5419	TL	979.2	5113	526.3	1129.0	568	27.1
531	1058	1263	-807.4	3354	TL	1009.2	2101	125.9	1132.3	314	27.8

Table 1: Results for the GRID2SQUARE instances.

The paper is currently under review, but a preprint is available as [RCH$^+$16].

4.10 Approximation of Ellipsoids Using Bounded Uncertainty Sets

We studied the problem of approximating ellipsoid uncertainty sets with bounded (Bertsimas-Sim-type) uncertainty sets. Robust linear programs with ellipsoid uncertainty lead to quadratically constrained programs, whereas robust linear programs with bounded uncertainty sets remain linear programs which are generally easier to solve. Hence, it can be beneficial to replace ellipsoid uncertainty sets with bounded uncertainty sets.

Notation:

- An *ellipsoid uncertainty set* is given by $\mathcal{E}(a^0, M) = \{x \mid (x - a^0)^T M(x - a^0) \leq 1\}$, where M is a positive semidefinite matrix and a^0 is called the center of the ellipsoid.
- A *bounded uncertainty set* is a polytope $\mathcal{U}(a^0, a, \Gamma) \subset \mathbb{R}^n$ that is characterized by $a^0, a \in \mathbb{R}^n$ and a budget parameter $\Gamma \in [0, n]$.

We considered two different inner approximation problems. The first problem is to find a bounded uncertainty set which sticks close to the ellipsoid such that a shrunk version of the ellipsoid is contained in it. The approximation is optimal if the required shrinking is minimal.

$$\begin{aligned} \max_{r,a,\Gamma} \; & r \\ \text{s.t. } & r\mathcal{E}(a^0, M) \subset \mathcal{U}(a^0, a, \Gamma) \subset \mathcal{E}(a^0, M) \end{aligned} \tag{AP-R}$$

In the second problem, we search for a bounded uncertainty set within the ellipsoid with maximum volume.

$$\begin{aligned} \max_{a,\Gamma} \; & \text{vol}(\mathcal{U}(a^0, a, \Gamma)) \\ \text{s.t. } & \mathcal{U}(a^0, a, \Gamma) \subset \mathcal{E}(a^0, M) \end{aligned} \tag{AP-V}$$

We derived explicit analytic formulas for the optimal solutions of these problems. To approximate also general ellipsoids we introduced the notion of rotated bounded uncertainty sets.

Theorem: Given a general ellipsoid $\mathcal{E}(a^0, M) \subset \mathbb{R}^n$ with $M = RDR^T$ and $D = diag\left(\frac{1}{d_1^2}, \ldots, \frac{1}{d_n^2}\right)$. The rotated bounded uncertainty set which gives the *best* approximation of this ellipsoid is described by

$$\begin{aligned} R_i^T(x - a^0) &\leq z_i \;\forall i \in [n] \\ -R_i^T(x - a^0) &\leq z_i \;\forall i \in [n] \\ z_i\sqrt{\lfloor\Gamma^*\rfloor + (\Gamma^* - \lfloor\Gamma^*\rfloor)^2} &\leq d_i \;\forall i \in [n] \\ \sum_{i=1}^{n} z_i \frac{\sqrt{\lfloor\Gamma^*\rfloor + (\Gamma^* - \lfloor\Gamma^*\rfloor)^2}}{d_i} &\leq \Gamma^*. \end{aligned}$$

For ratio approximation Γ^* is set to $\sqrt{n}$. For volume approximation Γ^* is set to $\Gamma^*(n) = \text{amin}_\Gamma(\lfloor\Gamma\rfloor + (\Gamma - \lfloor\Gamma\rfloor)^2)^{-\frac{n}{2}} \frac{1}{n!} \sum_{k=0}^{\lfloor\Gamma\rfloor}(-1)^k \binom{n}{k} (\Gamma - k)^n$.

To prove the benefit of replacing ellipsoids with bounded uncertainty sets we did the following experiment. We considered the problem $\mathcal{P}$ of finding the robust shortest path in a graph where the edge costs are affected by an ellipsoidal uncertainty set. We used the derived formula to approximate the ellipsoidal uncertainty set with a rotated bounded uncertainty set. The resulting problems are denoted by $\mathcal{P}'$. It turned out that the optimal solutions of $\mathcal{P}'$ are very close or even exactly the optimal solutions of the original problem but can be solved considerably faster as seen in Figure 9.

A preprint is available as [Cha16].

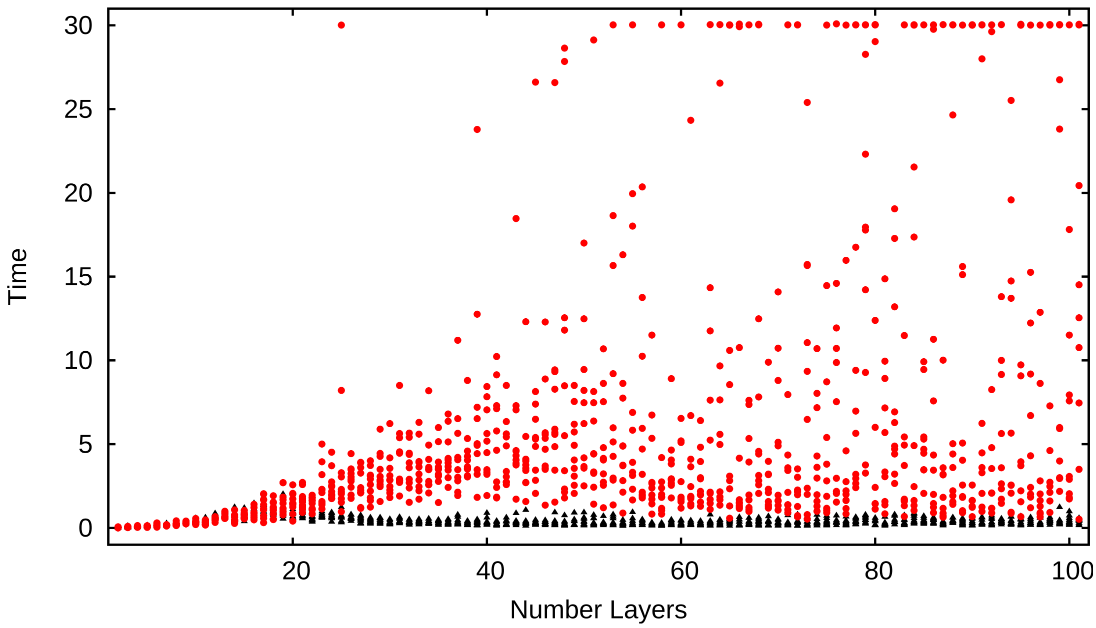


Figure 9: The red dots represent the time to solve $\mathcal{P}$ and the black triangles the time to solve $\mathcal{P}'$. The computation time is given in seconds. The time limit is set to 30 seconds.

4.11 Min-Max Regret Problems with Ellipsoidal Uncertainty Sets

We considered general combinatorial optimization problems of the form

$$\min\{c^T x : x \in \mathcal{X} \subseteq \{0,1\}^n\}$$

where the objective vector c is unknown, and coming from a set $\mathcal{U}$ of possible realizations. The robust objective function in the sense of regret problems is then

$$Reg(x,\mathcal{U}) = \max\{c^T x - opt(c) : c \in \mathcal{U}\}$$

with $opt(c)$ being the optimal objective value of the original problem with objective function c. This problem has been extensively analyzed for finite and hyperbox uncertainty sets. For the first time, we considered the use of ellipsoidal uncertainty sets in this setting. We distinguish between general ellipsoids and axis-parallel ellipsoids. We considered two different combinatorial optimization problems: The unconstrained combinatorial optimization problem where

$\mathcal{X} = \{0,1\}^n$, and the classic shortest path problem. The obtained complexity results are summarized in Tables 2 and 3.

	Interval	Finite	Axis-Parallel Ellipsoid	General Ellipsoid
Eval	P*	P*	NPC*	NPC*
Solve	P*	NPC*	P*	NPC*

Table 2: Overview of the different complexity results of the minmax regret unconstrained combinatorial problem.

	Interval	Axis-Parallel Ellipsoid	General Ellipsoid	Finite
Eval	P	NPC*	NPC*	P
Solve	NPC	NPC*	NPC*	NPC

Table 3: Overview of the different complexity results of the minmax regret shortest path problem.

After the theoretical investigation of these problems we developed two different ways to reformulate the minmax regret problem via a scenario relaxation procedure, resulting in exact, general solution approaches. These algorithms are compared in computational experiments.

The research on this topic has been published as [CG16b], where details can be found.

4.12 Ranking Robustness and its Application to Evacuation Planning

A new approach to robust optimization was developed, which is based on a preference ranking of solutions. The basic idea is motivated from decision making in emergency management, where precise "objective values" may not be very meaningful, but instead, a rough classification of "good" and "bad" solutions in each scenario can be given. A robust solution in the sense of ranking robustness is one for which the worst-case ranking is as good as possible.

More formally, we write

$$P(c) \quad \min\{f(x,c) \ : \ x \in \mathcal{X}\}, \quad c \in \mathcal{U} \tag{1}$$

where $\mathcal{X}$ denotes the set of feasible solutions, and $\mathcal{U}$ a set of possible scenarios, the so-called uncertainty set. Let a priority list $\mathcal{S}(c)$ be given for every scenario $c \in \mathcal{U}$, and let $K \in \mathbb{N}$. Then we denote with

$$\mathcal{X}^K(c) := \bigcup_{i \leq K} S_i(c) \tag{2}$$

the set of feasible solutions with preference at most K in scenario c, and with

$$\mathcal{X}^K := \bigcap_{c \in \mathcal{U}} \mathcal{X}^K(c). \quad (3)$$

We say a solution $x \in \mathcal{X}$ is K-ranking robust if $x \in \mathcal{X}^K$. The (general) ranking robustness problem (RR) consists in finding $K^* := \min\{K \in \mathbb{N}_+ : \mathcal{X}^K \neq \emptyset\}$, i.e. the smallest K for which a K-ranking robust solution exists.

As a numerical example, consider the following minimization problem with two scenarios c_1 and c_2, and three solutions A, B, and C. The objective values are given in Table 4.

	A	B	C
c_1	50	21	10
c_2	100	105	110

Table 4: Objective values of an example problem.

Solution A has the best worst-case performance, and is the optimal solution to class min-max robustness. However, it ignores the poor performance of A compared to B and C in scenario c_1. Solution C has the smallest maximum regret, and is the optimal solution to min-max regret robustness. Solution B is the second-best solution in every scenario, and is thus also interesting as a compromise solution from a practical perspective (while both A and C can be the worst choices in one of the scenarios, respectively).

We analyzed ranking robustness for two special cases, which we called solution ranking and objective ranking. Problem complexities and solution algorithms are presented. In a computational example, we considered Kulmbach in the south east of Germany, and calculated robust evacuation paths for different flooding scenarios. Our experiments indicated that ranking robust solutions give an interesting trade-off between robustness and performance, but require high computation times. Further research will concentrate on more efficient exact solution methods, and possibly heuristic algorithms.

The research on this topic has been published as [GHK16], where details can be found.

5 Project Dissemination Overview

Accepted papers:

- A. Chassein, M. Goerigk. *Minmax regret combinatorial optimization problems with ellipsoidal uncertainty sets.* To appear in European Journal of Operational Research, 2016, doi:10.1016/j.ejor.2016.10.055

- M. Goerigk, H. W. Hamacher, A. Kinscherff. *Ranking Robustness and its Application to Evacuation Planning.* To appear in European Journal of Operational Research, 2016, doi:10.1016/j.ejor.2016.05.037
- A. Chassein and M. Goerigk. *Performance analysis in robust optimization.* In E. Grigoroudis M. Doumpos, C. Zopounidis, editor, Robustness Analysis in Decision Aiding, Optimization, and Analytics, volume 241 of International Series in Operation Research & Management Science, pages 145170. Springer International Publishing, 2016.
- A. Chassein, M. Goerigk. *A Bicriteria Approach to Robust Optimization.* Computers and Operations Research, Volume 66, February 2016, Pages 181-189
- A. Chassein and M. Goerigk. *On the recoverable robust traveling salesman problem.* Optimization Letters, 10(7):181189, 2016.
- A. Chassein, M.Goerigk. *A new bound for the midpoint solution in minmax regret optimization with an application to the robust shortest path problem.* European Journal of Operational Research, Volume 224, Issue 3, 1 August 2015, Pages 739-747
- A. Chassein, M. Goerigk. *Alternative formulations for the ordered weighted averaging objective.* Information Processing Letters, Volume 115, Issues 6-8, June-August 2015, Pages 604-608
- M. Goerigk, H. W. Hamacher. *Optimisation Models to Enhance Resilience in Evacuation Planning.* Civil Engineering and Environmental Systems. Volume 32, Issue 1-2, 2015.

Submitted / Preprints:

- B. Rostami, A. Chassein, M. Hopf, D. Frey, C. Buchheim, F. Malucelli, M. Goerigk. *The Quadratic Shortest Path Problem: Complexity, Approximability, and Solution Methods.* Submitted to Mathematical Programming, downloadable from semanticscholar.org under `https://pdfs.semanticscholar.org/4f9a/eda15c0139cfb64ecd90f842a9185568bc89.pdf`
- A. Chassein. *Approximation of Ellipsoids Using Bounded Uncertainty Sets.* Preprint University of Kaiserslautern, downloadable from the KLUEDO server under `https://kluedo.ub.uni-kl.de/frontdoor/index/index/docId/4344`

- A. Chassein, M. Goerigk. *Robust Geometric Programming is co-NP hard.* Preprint University of Kaiserslautern, downloadable from the KLUEDO server under `https://kluedo.ub.uni-kl.de/frontdoor/index/index/docId/3938`

In preparation (working titles):

- A. Chassein, *Robust Optimization: Complexity and Solution Methods* PhD thesis. (Under Review)

Presentations:

- M. Goerigk. *Solving Combinatorial Min-Max Regret Problems with Non-Interval Uncertainty Sets.* International Conference on Operations Research, Hamburg 08/2016.
- A. Chassein. *Approximation of Ellipsoids Using Bounded Uncertainty Sets.* 28th European Conference on Operational Research (EURO), Poznan 07/2016.
- M. Goerigk. *First Ideas on Inverse Robust Optimization Problems.* 28th European Conference on Operational Research (EURO), Poznan 07/2016.
- A. Chassein. *An Introduction to Robust Optimization.* Workshop on Stochastic Approaches in Engineering and Financial Mathematics, Hanover, 03/2016.
- A. Chassein. *Reliable Shortest Path Problems.* Oberseminar TU Dortmund, Dortmund, 09/2015.
- A. Chassein. *A Bicriteria Approach to Robust Optimization.* International Conference on Operations Research, Vienna, 09/2015.
- M. Goerigk. *Flood Evacuation Planning Using a New Approach to Robustness.* 27th European Conference on Operational Research (EURO), Glasgow 07/2015.
- A. Chassein. *Reliable Shortest Path Problems.* 27th European Conference on Operational Research (EURO), Glasgow 07/2015.
- M. Goerigk. *A Bicriteria Approach to Robust Optimization.* Indo-German Workshop on Algorithms, Kolkata, 03/2015.
- M. Goerigk. *Challenges in Robust Optimization.* Invited Seminar at the University of Osnabrck, 11/2014.
- A. Chassein. *Minmax Regret: Improved Analysis for the Midpoint Solution.* International Conference on Operations Research, Aachen, 09/2014.

- A. Chassein. *On Comparing Robustness Approaches for Timetabling.* Conference of the International Federation of Operational Research Societies (IFORS), Barcelona, 07/2014.

Lectures:

- M. Goerigk. *Robust Optimization.* Summer Semester 2015, TU Kaiserslautern.
- H. W. Hamacher. *Robust Optimization and Multicriteria Optimization.* Winter Semester 2014/15, TU Kaiserslautern.

6 Conclusions

All problems considered in the project period were investigated in two different ways. We started with a theoretically analysis of the computational complexity of the problem. Followed by an implementation of exact, approximation, or heuristic algorithms to solve the problem. Some highlights of the obtained results are listed in the following:

- We improved the formulation of the ordered weighted averaging problem improving the computation time of the method.
- We found a very efficient way to define an aposteriori approximation guarantee for the midpoint solution for min max regret problems.
- We developed the concept of the scenario curve to provide a fair evaluation between different robust solutions.
- We give an extensive analysis of the computational complexity of the quadratic shortest path problem and several variants.
- We define efficient algorithms for the reliable shortest path problems on graphs with special structure.
- We present a bicriteria approach, the AC-WC curve, to compare robust solution concepts.
- We made use of multi-criteria approaches in robust optimization to develop new, efficient solution algorithms.

Most of the results obtained in this project will be summarized in the PhD thesis of André Chassein.

References

[ABV05] H. Aissi, C. Bazgan, and D. Vanderpooten. Complexity of the min–max and min–max regret assignment problems. *Operations research letters*, 33(6):634–640, 2005.

[ABV09] H. Aissi, C. Bazgan, and D. Vanderpooten. Minmax and minmax regret versions of combinatorial optimization problems: A survey. *European Journal of Operational Research*, 197(2):427 – 438, 2009.

[AL05] I. Averbakh and V. Lebedev. On the complexity of minmax regret linear programming. *European Journal of Operational Research*, 160(1):227 – 231, 2005.

[Ave01] I. Averbakh. On the complexity of a class of combinatorial optimization problems with uncertainty. *Mathematical Programming*, 90(2):263–272, 2001.

[BBC11] D. Bertsimas, D. Brown, and C. Caramanis. Theory and applications of robust optimization. *SIAM Review*, 53(3):464–501, 2011.

[BS04] D. Bertsimas and M. Sim. The price of robustness. *Operations Research*, 52(1):35–53, 2004.

[BTGN09] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust Optimization*. Princeton University Press, Princeton and Oxford, 2009.

[BTN98] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

[BTN99] A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25:1–13, 1999.

[CG14] A. Chassein and M. Goerigk. Robust geometric programming is co-np hard. Technical report, Arbeitsgruppe Optimierung, Technische Universitt Kaiserslautern, 2014.

[CG15a] A. Chassein and M. Goerigk. Alternative formulations for the ordered weighted averaging objective. *Information Processing Letters*, 115(6-8):604–608, 2015.

[CG15b] A. Chassein and M. Goerigk. A new bound for the midpoint solution in minmax regret optimization with an application to the robust shortest path problem. *European Journal of Operational Research*, 244(3):739–747, 2015.

[CG16a] A. Chassein and M. Goerigk. A bicriteria approach to robust optimization. *Computers and Operations Research*, 66(0):181–189, 2016.

[CG16b] A. Chassein and M. Goerigk. Minmax regret combinatorial optimization problems with ellipsoidal uncertainty sets. *European Journal of Operational Research*, To appear:–, 2016. To appear.

[CG16c] A. Chassein and M. Goerigk. On the recoverable robust traveling salesman problem. *Optimization Letters*, 10(7):181–189, 2016.

[CG16d] A. Chassein and M. Goerigk. Performance analysis in robust optimization. In E. Grigoroudis M. Doumpos, C. Zopounidis, editor, *Robustness Analysis in Decision Aiding, Optimization, and Analytics*, volume 241 of *International Series in Operation Research & Management Science*, pages 145–170. Springer International Publishing, 2016.

[Cha16] A. Chassein. Approximation of ellipsoids using bounded uncertainty sets. Technical report, Arbeitsgruppe Optimierung, Technische Universitt Kaiserslautern, 2016.

[Coo12] William Cook. *In pursuit of the traveling salesman: mathematics at the limits of computation.* Princeton University Press, 2012.

[GHK16] M. Goerigk, H. W. Hamacher, and A. Kinscherff. Ranking robustness and its application to evacuation planning. *European Journal of Operational Research*, pages –, 2016. To appear.

[GHMH+13] Marc Goerigk, Sacha Heße, Matthias Müller-Hannemann, Marie Schmidt, Anita Schöbel, et al. Recoverable robust timetable information. In *ATMOS-13th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems-2013*, volume 33, pages 1–14, 2013.

[Goe14] M. Goerigk. A note on upper bounds to the robust knapsack problem with discrete scenarios. *Annals of Operations Research*, 223(1):461–469, 2014.

[GS16] M. Goerigk and A. Schöbel. Algorithm engineering in robust optimization. In L. Kliemann and P. Sanders, editors, *Algorithm Engineering: Selected Results and Surveys*, volume 9220 of *Lecture Notes in Computer Science*, pages 245–279. Springer International Publishing, 2016.

[HKB08] Kan-Lin Hsiung, Seung-Jean Kim, and Stephen Boyd. Tractable approximate robust geometric programming. *Optimization and Engineering*, 9(2):95–118, 2008.

[Iid99] Hiroshi Iida. A note on the max-min 0-1 knapsack problem. *Journal of Combinatorial Optimization*, 3(1):89–94, 1999.

[IS95] M. Inuiguchi and M. Sakawa. Minimax regret solution to linear programming problems with an interval objective function. *European Journal of Operational Research*, 86(3):526 – 536, 1995.

[KY97] P. Kouvelis and G. Yu. *Robust Discrete Optimization and Its Applications*. Kluwer Academic Publishers, 1997.

[LLMS09] C. Liebchen, M. Lübbecke, R. H. Möhring, and S. Stiller. The concept of recoverable robustness, linear programming recovery, and railway applications. In R. K. Ahuja, R.H. Möhring, and C.D. Zaroliagis, editors, *Robust and online large-scale optimization*, volume 5868 of *Lecture Note on Computer Science*, pages 1–27. Springer, 2009.

[MGD04] R. Montemanni, L.M. Gambardella, and A.V. Donati. A branch and bound algorithm for the robust shortest path problem with interval data. *Operations Research Letters*, 32(3):225 – 232, 2004.

[MP13] M. Monaci and U. Pferschy. On the robust knapsack problem. *SIAM OPT*, 23(4):1956–1982, 2013.

[OŚ03] Włodzimierz Ogryczak and Tomasz Śliwiński. On solving linear programs with the ordered weighted averaging objective. *European Journal of Operational Research*, 148(1):80–91, 2003.

[RCH+16] B. Rostami, A. Chassein, M. Hopf, D. Frey, C. Buchheim, F. Malucelli, and M. Goerigk. The quadratic shortest path problem: Complexity, approximability, and solution methods. Technical report, Optimization Online, 2016.

[TYK08] Fumiaki Taniguchi, Takeo Yamada, and Seiji Kataoka. Heuristic and exact algorithms for the maxmin optimization of the multi-scenario knapsack problem. *Computers & Operations Research*, 35(6):2034 – 2048, 2008.

List of Symbols, Abbreviations, and Acronyms

TARCMO	Theory and Algorithms for Robust, Combinatorial, Multicriteria Optimization
$\mathcal{U}$	Uncertainty set
$f(x, \xi)$	Uncertain objective function
$F(x, \xi)$	Uncertain constraint function
$\mathcal{F}(\xi)$	Set of feasible solutions in scenario $\xi \in \mathcal{U}$
$\mathcal{X}$	Feasible set not affected by uncertainty